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ON THE CREEP BUCKLING OF SHELLS

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Abstract—In the present work the buckling of shells is considered in which the material exhibits the property of creep.

The authors suppose that after loading the shell does not lose stability. Under the action of load the shell is bending. It takes place during time of change of shell curvature and internal forces at the expense of original deflection and deformation of creep of material. This process of development leads during the time to the instability of initial form of equilibrium. In this way critical time is determined.

At first the authors find the stress and strain state of the shell before buckling, and then investigate the possibility of change of the equilibrium form.

It is given physical and geometrical nonlinear formulation of the task; for prebuckling state physical correlation are linearized. Nonlinear system of differential equation, which describes the state of the shell before buckling, is solved by step method. A system of differential equation, which describes neutral equilibrium, is linear.

Total theory is illustrated for the case of an axially compressed circular cylindrical shell and for the case of a line radially compressed load on a circular cylindrical shell. Essentially, Bubnov's method is used here. Note that in the first task the critical time is finite and it corresponds to the time of the initial transition of the axisymmetrical form of the shell in unsymmetrical form.

1. INTRODUCTION

IN recent years there have appeared a great number of works concerned with the problem of creep buckling. Since different bodies behave differently during creep and their behaviour is described by various rheological equations, there exist a number of fundamentally different approaches to the analysis of stability of systems under creep [1–6]. For a linearly viscoelastic bar, the work of Rzhanitsyn [1] who studied the nature of the disturbed motion over an infinitely long period of time is noteworthy. The bar is assumed to be perfectly straight before loading.

In the case of a creeping material obeying the strain–hardening hypothesis such an investigation becomes rather difficult owing to a non-linear stress–strain relation. Rabotnov and Shesterikov [2] overcome this difficulty by linearizing the physical relations with respect to the basic state corresponding to the undisturbed motion and analyze the beginning of the disturbed motion over an infinitely short period of time. The critical time is taken to be that from which the development of disturbances occurs with positive velocity. Such an approach is, to some extent, limited and, as subsequent investigations [3] show, its agreement with experimental results is very difficult to achieve.

A different method of attack is found in the American works [5] (see also [6]). Here the development of the initial deflection of a system in time is studied. The initial deflection of a system exists regardless of its heating and loading conditions and characterizes the imperfections of the shape of the structure due to manufacturing and other causes. The critical time is then defined by the instant at which the deflection becomes infinite or the instant at which the rate of change of the deflection becomes infinite. For a bar, this problem is solved from the point of view of the geometrically linear theory using non-linear physical stress–strain relations. In the case of unlimited creep such an approach to the solution of the problem appears to be most worth-while and it can be extended in a natural way to the solution of the problem of shell buckling [7, 8].

In the analysis of shell buckling, however, the consideration of geometrical non-linearity is of prime importance. This was shown in [9, 19]. In [10], in the derivation of equations expressing the development of deflections in time, the stress–strain relations are linearized and primary emphasis is placed upon the consideration of geometrical non-linearity.

Considering the variation of deflections of a shell in time on the basis of geometrically non-linear equations it may be observed that at a certain instant of time $t = t_*$ there is a change in equilibrium configurations accompanied by the snap-through buckling of the shell (see Fig. 1, where ζ is the characteristic deflection). As a result the critical time is

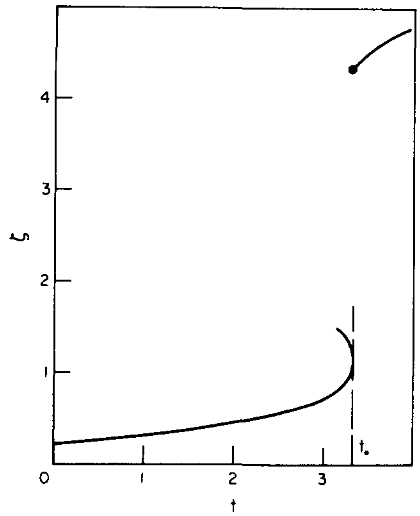

FIG. 1

defined by the moment of snap-through buckling. The calculation of this time involves the integration of a system of non-linear integral equations expressing the development of deflections during creep.

We may mention a series of fundamentally different approaches to the solution of the problem discussed, but none of these can claim generality. This is due to the fact that when choosing the statement of the problem and the stability criterion various authors used each time different premises based most frequently on the consideration of specific circumstances in the formulation of stress–strain equations, while the consideration of creep does not lead to a radical alteration in the understanding of the stability concept and the methods of solution which were formed in the analysis of stability of elastic and elastic–plastic systems. As was shown in [11] and [12], only the design scheme is changed and refined, these modifications being substantial in that part which is associated with the determination of stresses and strains in the undisturbed state of a system, while the equations of neutral equilibrium written in terms of instantaneous increments of force and displacement are of the same form as for elastic systems. In these works, the possibilities are discussed for applying the static method of analysis of shell stability after Euler, with the equations of neutral equilibrium defining the moment of bifurcation buckling written so as to take account of additional curvatures and forces due to creep. Below this statement of the problem is developed further, the derivation of basic equations is given and the results of solutions of some problems are presented.

2. STATEMENT OF THE PROBLEM

Consider a thin homogeneous shell with the principal radii of curvature R_1, and R_2 and thickness h loaded so as to cause compressive forces in the middle surface of the shell. If the load parameters do not exceed a certain critical value, the original configuration of equilibrium is stable after the load is applied. Further, the shell bends and additional forces are developed during creep. This process of development of additional curvatures and additional forces may be so pronounced that after a certain finite interval of time the original configuration of equilibrium becomes unstable, with the result that there occurs a change in equilibrium configurations accompanied by snap-through buckling and the formation of a typical buckle pattern.

With this mechanism of instability it seems natural to separate the solution of the problem into two stages. The first stage of solution of the problem consists in determining the state of stress and strain of the shell in the prebuckling stage of the process of creep before bifurcation buckling occurs. After the stresses and strains in the shell are found, it is necessary to ascertain the possibility of a change in equilibrium configurations during creep. This problem reduces to that of finding non-zero solutions of the homogeneous system of linear differential equations of neutral equilibrium with homogeneous boundary conditions written in terms of the variations of stresses and deflection.

We now proceed to the derivation of the equations of the prebuckling state defining the state of stress and strain of the shell in the process of creep and of the equations of neutral equilibrium which provide means of analyzing the stability of this process.

3. THE EQUATIONS OF THE PREBUCKLING STATE

The only difference between the mathematical formulation of the problem of determining the state of stress and strain in creeping bodies and the usual statement of the problem for an elastic body lies in the choice of the form of stress–strain relations while the equilibrium equations and the relations between strains and displacements are the same as for an elastic body. Here use is made of non-linear stress–strain relations and non-linear equilibrium equations and the solution of the problem is developed on the basis of the Sanders–McComb–Schlechte variational theorem [13].

According to the Sanders–McComb–Schlechte variational theorem the problem of determining the state of stress and strain in creeping solids reduces to the solution of the variational equation

$$\delta\mathcal{J} = 0 \tag{3.1}$$

where the functional $\mathscr{J}$ is expressed as

$$\mathscr{J} = \int_V [\dot{\varepsilon}_{ij}\dot{\sigma}_{ij} + \tfrac{1}{2}\dot{u}_{k,i}\dot{u}_{k,j}\sigma_{ij} - \tfrac{1}{2}(\dot{\varepsilon}^e_{ij} + 2\dot{p}_{ij})\dot{\sigma}_{ij}]\,\mathrm{d}V - \int_{S_s} \dot{T}^*_i \dot{u}_i\,\mathrm{d}S - \int_{S_d} (\dot{u}_i - \dot{u}^*_i)\dot{T}_i\,\mathrm{d}S \quad (3.2)$$

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \tfrac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i} + \dot{u}_{k,i}u_{k,j} + u_{k,i}\dot{u}_{k,j}) \\ T_i &= \sigma_{jk}n_j(\delta_{ik} + u_{i,k}). \end{aligned} \quad (3.3)$$

Here ε_{ij} are the components of the total strain and are expressed in terms of displacements u_i ($i = 1, 2, 3$), ε^e_{ij} is the elastic–plastic part of the total strain and is expressed in terms of the stress, $\dot{p}_{ij}$ are the components of creep stain rates, $\dot{u}^*_i$, $\dot{T}^*_i$ are given quantities on the surface, S_s is a portion of the surface where stresses are prescribed, S_d is a portion of the surface where displacements are prescribed.

It should be noted that the components of creep strain rates $\dot{p}_{ij}$ are assumed to be dependent only on the stress deviator and the second invariant of the stress tensor and independent of their rates. Hence, in setting up the condition for the stationarity of functional (3.2) we have $\delta\dot{p}_{ij} = 0$ since only the rates $\dot{u}_i$ and the time derivatives $\dot{\sigma}_{ij}$ are varied according to the conditions of the theorem.

If we make some simplifying assumptions regarding the dependence of displacements and stresses on the co-ordinate z measured along the normal to the middle surface, then the integration with respect to z in the integral $\mathscr{J}$ can be performed and this in turn enables one to pass to a two-dimensional problem similar to elastic shell problems. A particular advantage of the generalized Sanders–McComb–Schlechte variational theorem lies in the fact that approximations for stresses and strains can be made independently. Hence, there is no need to invert the stress–strain relations in order to obtain a suitable approximation to the stress distribution.

Regarding the nature of displacement distribution it is possible to make the same simplifying assumptions as for thin elastic shells. Thus, accepting the Kirchhoff–Love hypotheses for strains in a shell we have

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i} + w_{,i}w_{,j}) - k_{ij}w \\ u_i = v_i - zw_{,i}; \quad k_{11} &= \frac{1}{R_1}; \quad k_{22} = \frac{1}{R_2}; \quad k_{12} = 0. \end{aligned} \quad (3.4)$$

Here v_i are the tangential displacements of points in the middle surface, w is the displacement along the normal (deflection of the shell). Accordingly, it is necessary also to simplify the expression for the functional $\mathscr{J}$ and to retain only the powers and products of the rates of change of slope in the second term of the volume integral, see equation (3.2).

As regards the nature of stress distribution across the thickness, there are various methods of approximation available and each of these methods leads to a system of differential equations different from the other ones. Thus, not all particular systems of differential equations can be called the equations of the theory of creep of shells. Therefore, further details of the application of the variational theorem will be illustrated by a simple example.

Let the relations between the strain rates $\dot{\varepsilon}^e_{ij}$, $\dot{p}_{ij}$ and the components of the stress deviator s_{ij} be

$$\dot{\varepsilon}^e_{ij} = \frac{3}{2E}\dot{s}_{ij}; \qquad \dot{p}_{ij} = \frac{3A}{2}\sigma_i^{m-1}s_{ij}. \quad (3.5)$$

Here σ_i is the intensity of normal stresses and so the summation convention is not extended to the symbol σ_i which, as usual, denotes a scalar quantity ($\sigma_i^2 = \frac{3}{2}s_{kn}s_{kn}$); A and m are constants.

We introduce a linear law of stress distribution across the thickness of a shell

$$\sigma_{ij} = \frac{1}{h}T_{ij}+\frac{12}{h^3}M_{ij}z \tag{3.6}$$

where T_{ij} are the forces per unit length referred to the middle surface, M_{ij} are the bending moments.

Substituting relations (3.4) through (3.6) in equation (3.2) and carrying out the integration with respect to z, retaining only terms of order not higher than 3, we obtain

$$\begin{aligned}\mathscr{J} = \int_S \Big\{ &[\tfrac{1}{2}(\dot{v}_{i,j}+\dot{v}_{j,i}+\dot{w}_{,i}w_{,j}+w_{,i}\dot{w}_{,j})-k_{ij}\dot{w}]\dot{T}_{ij}-\dot{M}_{ij}\dot{w}_{,ij}+\tfrac{1}{2}\dot{w}_{,i}\dot{w}_{,j}T_{ij}\\ &-\frac{3}{4Eh}\dot{T}_{ij}\dot{N}_{ij}-\frac{9}{Eh^3}\dot{M}_{ij}\dot{G}_{ij}-\dot{q}\dot{w}-\frac{3A}{2h}[(A_1N_{ij}+B_1G_{ij})\dot{T}_{ij}\\ &+(C_1G_{ij}+B_1N_{ij})\dot{M}_{ij}]\Big\}\,\mathrm{d}S-\mathscr{J}_l^*\end{aligned} \tag{3.7}$$

$$N_{ij} = T_{ij}-\tfrac{1}{3}\delta_{ij}T_{kk}; \qquad G_{ij} = M_{ij}-\tfrac{1}{3}\delta_{ij}M_{kk}$$

$$i, j, k = 1, 2.$$

Here $\mathscr{J}_l^*$ are contour integrals, q is the lateral pressure, A_1, B_1, and C_1 stand for the expressions

$$\begin{aligned}A_1 &= \left(\frac{T}{h}\right)^{m-1}+\frac{6(m-1)}{h^4}M^2\left(\frac{T}{h}\right)^{m-3}+\frac{6(m-1)(m-3)}{h^6}H^4\left(\frac{T}{h}\right)^{m-5}\\ B_1 &= \frac{12(m-1)}{h^4}H^2\left(\frac{T}{h}\right)^{m-3}; \qquad C_1 = \frac{12}{h^2}\left(\frac{T}{h}\right)^{m-1};\\ T^2 &= \tfrac{3}{2}T_{ij}N_{ij}; \qquad M^2 = \tfrac{3}{2}M_{ij}G_{ij}; \qquad H^2 = \tfrac{3}{2}T_{ij}G_{ij}.\end{aligned} \tag{3.8}$$

If we now introduce the force function F for T_{ij} and use the transformation of the integral

$$\tfrac{1}{2}\int_S(\dot{v}_{i,j}+\dot{v}_{j,i})\dot{T}_{ij}\,\mathrm{d}S = \int_l \dot{v}_i n_j\dot{T}_{ij}\,\mathrm{d}l-\int_S \dot{v}_i\dot{T}_{ij,j}\,\mathrm{d}S$$

the expression for function (3.7) is greatly simplified and becomes

$$\begin{aligned}\mathscr{J} = \int_S \Big\{&(\tfrac{1}{2}\dot{w}_{,i}w_{,j}+\tfrac{1}{2}w_{,i}\dot{w}_{,j}-k_{ij}\dot{w})\dot{T}_{ij}-\dot{w}_{,ij}\dot{M}_{ij}+\tfrac{1}{2}\dot{w}_{,i}\dot{w}_{,j}T_{ij}-\frac{3}{4Eh}\dot{T}_{ij}\dot{N}_{ij}\\ &-\frac{9}{Eh^3}\dot{M}_{ij}\dot{G}_{ij}-\dot{q}\dot{w}-\frac{3A}{2h}[(A_1N_{ij}+B_1G_{ij})\dot{T}_{ij}+(C_1G_{ij}+B_1N_{ij})\dot{M}_{ij}]\Big\}\,\mathrm{d}S-\mathscr{J}_l.\end{aligned} \tag{3.9}$$

The expression for $\mathscr{J}_l$ incorporates contour integrals

$$\mathscr{J}_l = \int_{l_s} \left[(\dot{T}_\nu^* - \dot{T}_\nu)\dot{v}_\nu + (\dot{T}_\tau^* - \dot{T}_\tau)\dot{v}_\tau - \dot{M}_\nu^* \dot{w}_{,\nu} + \dot{w}\frac{\mathrm{d}}{\mathrm{d}t}(Q^* + M_{\tau,\tau}^* + T_\nu^* w_{,\nu}^* + T_\tau^* w_{,\tau}^*) \right] \mathrm{d}l$$

$$- \int_{l_s} \left[\dot{T}_\nu \dot{v}_\nu^* + \dot{T}_\tau^* + \dot{M}_\nu(\dot{w}_{,\nu} - \dot{w}_{,\nu}^*) - (\dot{w} - \dot{w}^*)\frac{\mathrm{d}}{\mathrm{d}t}(Q + M_{\tau,\tau} + T_\nu w_{,\nu} + T_\tau w_{,\tau}) \right] \mathrm{d}l \quad (3.10)$$

$$T_\nu = T_{ij} n_i n_j; \qquad M_\nu = M_{ij} n_i n_j;$$

$$T_\tau = (T_{22} - T_{11}) n_1 n_2 + T_{12}(n_1^2 - n_2^2);$$

$$M_\tau = (M_{22} - M_{11}) n_1 n_2 + M_{12}(n_1^2 - n_2^2).$$

Here n_i are the components of the unit vector normal to the shell contour, v_ν, v_τ are the displacements of points in the middle surface along the normal and the tangent to the shell contour, respectively.

Thus, as a result of the above transformation the integrand of functional (3.9) no longer depends on the tangential displacements v_1, v_2, and the forces T_{ij} are expressed in terms of the force function F

$$T_{ij} = (\delta_{ij}\nabla^2 F - F_{,ij});$$
$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}; \qquad x_1 = x; \qquad x_2 = y. \quad (3.11)$$

In setting up the condition for the stationarity of functional (3.9) the independent functions are taken to be the deflection w, the moments M_{ij} and the force function F. Then the time derivatives of these quantities only are varied.

The solution of the variational problem is carried on in the usual way. Specifying the dependence of the unknown functions on the co-ordinates x, y, we perform all the integrations necessary for the finding of $\mathscr{J}$. Setting the variation of the expression so obtained equal to zero gives a system of ordinary differential equations in which the independent variable is the time. The integration of this system of equations enables one to determine the dependence of stresses and strains in the shell on the time t.

4. LINEARIZED CREEP EQUATIONS FOR SHELLS

In the preceding section we reduced the problem of determining stresses and strains to the solution of a variational problem which is essentially non-linear. The non-linearity of the variational problem is due to the non-linearity of geometrical and physical relations (3.4) and (3.5). However, in cases where the basic state of stress corresponding to the membrane state of a smooth shell plays the predominant role during creep and the actual state of stress departs insignificantly from the basic one over a certain period of time up to the moment of bifurcation buckling, it is possible to linearize the physical relations (3.5). As a result the problem is greatly simplified and can readily be solved for any creep law.

Thus, we isolate the basic membrane state of stress σ_{ij}^0 from the actual state of stress σ_{ij}. Let σ_{ij}^+ denote the increments of stress due to the flexure of the shell which indicate the departure of the actual state of stress from the basic one. Equation (3.5) can now be represented as

$$\begin{aligned}\dot{p}^0_{ij}+\dot{p}^+_{ij} &= \frac{3A}{2}(\sigma^0_i+\sigma^+_i)^{m-1}(s^0_{ij}+s^+_{ij})\\ \dot{\varepsilon}^{e0}_{ij}+\dot{\varepsilon}^{e+}_{ij} &= \frac{3}{2E}(\dot{s}^0_{ij}+\dot{s}^+_{ij}).\end{aligned} \tag{4.1}$$

Retaining only linear terms in the increments on the right-hand side of the first equation, we arrive at the following equations

$$\begin{aligned}\dot{p}^+_{ij} &= \left(\frac{3A\sigma^0_i}{2}\right)^{m-1}\left[s^+_{ij}+\frac{(m-1)s^0_{ij}}{\sigma^0_i}\sigma^+_i\right]\\ \dot{\varepsilon}^{e+}_{ij} &= \frac{3}{2E}\dot{s}^+_{ij}.\end{aligned} \tag{4.2}$$

Thus, total strain rate increments $\dot{\varepsilon}^+_{ij}$ ($\dot{\varepsilon}^+_{ij} = \dot{\varepsilon}^{e+}_{ij}+\dot{p}^+_{ij}$) are related to stress increments by the linear equations

$$\dot{\varepsilon}^+_{ij} = \frac{3}{2E}\dot{s}^+_{ij}+\left(\frac{3A\sigma^0_i}{2}\right)^{m-1}\left[s^+_{ij}+\frac{(m-1)s^0_{ij}}{\sigma^0_i}\sigma^+_i\right].$$

Inverting these equations for stress increments, we obtain the following relations

$$\begin{aligned}s^+_{ij} &= \frac{2E}{3}\left(\varepsilon^+_{ij}-\int_0^\tau e^{\xi-\tau}\varepsilon^+_{ij}\,d\xi\right)+E(\alpha_{ij}-\tfrac{1}{3}\delta_{ij}\alpha_{kk})\int_0^\tau(e^{\xi-\tau}+m\,e^{m(\xi-\tau)})\alpha_{kn}\varepsilon^+_{kn}\,d\xi\\ \alpha_{ij} &= \sigma^0_{ij}/\sigma^0_i; \qquad \tau = E\int_0^t(A\sigma^0_i)^{m-1}\,dt\end{aligned} \tag{4.3}$$

where τ is a non-dimensional time parameter. Total strain increments in turn are expressed in terms of displacements

$$\begin{aligned}\varepsilon^+_{ij} &= \tfrac{1}{2}(u^+_{i,j}+u^+_{j,i}+w_{,i}w_{,j}-w^0_{,i}w^0_{,j})-k_{ij}(w-w^0)\\ u^+_i &= t^+_i-z(w_{,i}-w^0_{,i}).\end{aligned} \tag{4.4}$$

Here w^0 is the initial deflection characterizing the initial imperfections of the middle surface shape, w is the total deflection in bending.

Expressing now the forces and moments in terms of displacements by the use of equations (4.3) and (4.4) and substituting them in the equilibrium equations and the compatibility condition, we arrive at a system of two equations for the deflection function w and the force function Φ, which, according to the canonized theory of shallow shells, are (see also [11])

$$\begin{aligned}&D\nabla^2\nabla^2(w-w^0)-h\sigma^0_i\Lambda w-L(w,\Phi)-k_{11}\Phi_{,22}-k_{22}\Phi_{,11}-\frac{3mD}{4}\int_0^\tau e^{m(\xi-\tau)}\Lambda^2(w-w^0)\,d\xi\\ &\qquad -D\int_0^\tau e^{\xi-\tau}(\nabla^2\nabla^2-\tfrac{3}{4}\Lambda^2)(w-w^0)\,d\xi = 0\\ &\nabla^2\nabla^2\Phi+\int_0^\tau[\nabla^2\nabla^2+(m-1)\Lambda^2_1]\Phi\,d\xi = -Eh[k_{11}(w_{,22}-w^0_{,22})+k_{22}(w_{,11}-w^0_{,11})\\ &\qquad +\tfrac{1}{2}L(w,w)-\tfrac{1}{2}L(w^0,w^0)].\end{aligned} \tag{4.5}$$

The differential operators Λ, Λ_1, L are defined as

$$\begin{aligned} \Lambda_1 &= (\alpha_{11}-\tfrac{1}{2}\alpha_{22})\frac{\partial^2}{\partial x_2^2}+(\alpha_{22}-\tfrac{1}{2}\alpha_{11})\frac{\partial^2}{\partial x_1^2}-3\alpha_{12}\frac{\partial^2}{\partial x_1\partial x_2} \\ \Lambda &= \alpha_{11}\frac{\partial^2}{\partial x_1^2}+\alpha_{22}\frac{\partial^2}{\partial x_2^2}+2\alpha_{12}\frac{\partial^2}{\partial x_1\partial x_2} \\ L(w,\Phi) &= w_{,11}\Phi_{,22}+w_{,22}\Phi_{,11}-2w_{,12}\Phi_{,12}. \end{aligned} \tag{4.6}$$

In contrast to the preceding section, the force function is introduced so that

$$T_{ij}^{+} = (\delta_{ij}\nabla^2\Phi-\Phi_{,ij}). \tag{4.7}$$

Accordingly, the total forces are expressed through the force function Φ as

$$T_{ij} = h\sigma_{ij}^{0}+(\delta_{ij}\nabla^2\Phi-\Phi_{,ij}). \tag{4.8}$$

In conclusion we note that equations (4.5) may also be developed by a variational procedure. For this purpose it is necessary to vary functional (3.9) and to linearize the system of equations so obtained with respect to the basic state. After some manipulation this system of equations reduces to (4.5).

5. THE EQUATIONS OF NEUTRAL EQUILIBRIUM

The solution of the variational equation (3.1), where the functional $\mathscr{J}$ is defined by expression (3.9), or the solution of the system of equations (4.5) characterizes the variation of the state of stress and strain in the shell during creep. Let the original configuration of equilibrium become unstable at a certain instant of time $\tau = \tau^*$ and neighbouring configurations of equilibrium be possible which are characterized by a deflection $w+\varepsilon w^*$ and a force function $\Phi+\varepsilon\Phi^*$ (ε is a small parameter), where the functions w^* and Φ^* represent the increments of deflection and force in a possible transition of the shell to neighbouring configurations of equilibrium. If we now write the equilibrium equations and the compatibility condition for neighbouring configurations of equilibrium and take into account that instantaneous increments of stress and strain are related by Hooke's law, after a simple transformation these equations reduce to a homogeneous system of equations in the functions w^* and Φ^*:

$$\begin{aligned} D\nabla^2\nabla^2 w^*-h\sigma_i^0\Lambda w-L(w^*,\Phi)-L(w,\Phi^*)-k_{11}\Phi^*_{,22}-k_{22}\Phi^*_{,11} &= 0 \\ \nabla^2\nabla^2\Phi^* &= -Eh[k_{11}w^*_{,22}+k_{22}w^*_{,11}+L(w^*,w)]. \end{aligned} \tag{5.1}$$

Here the functions $w(x_1, x_2, \tau)$ and $\Phi(x_1, x_2, \tau)$ characterize the prebuckling state of the shell and are determined by the solution of the variational problem or by the solution of equations (4.5).

Thus, the problem of bifurcation buckling of a shell during creep reduces to the determination of a time parameter $\tau = \tau^*$ at which the system of equations (5.1), homogeneous in w^* and Φ^* with homogeneous boundary conditions, has non-zero solutions.

6. A METHOD OF SOLVING THE EQUATIONS OF THE PREBUCKLING STATE

We shall discuss the question of integrating non-linear equations by consideration of the system of equations (4.5). The system of equations (4.5) has been derived by linearizing the physical relations between stress and strain; nevertheless this system of equations is non-linear. The non-linearity of equations (4.5) is due to the use of the non-linear geometrical relations (4.4). Hence, the direct integration of the above equations involves considerable mathematical difficulties. However, the integration of these equations with respect to time can be performed by the step-by-step method and it is found convenient to apply the procedure of successive linearization of equations (4.5).

We shall say that the dependence of the unknown functions w, Φ on the time τ is known if the values of these functions are known for a series of consecutive values τ_k ($k = 0, 1, 2, \dots$) of the time parameter τ. On this understanding the problem reduces to the successive determination of values $w_k(x_1, x_2)$ and $\Phi_k(x_1, x_2)$ discrete in time.

Suppose that for a certain instant of time $\tau = \tau_k$ we know the solution of the system of equations (4.5). Denote it by $w_k(x_1, x_2)$, $\Phi_k(x_1, x_2)$. Then, for the succeeding instant of time $\tau = \tau_{k+1}$ the solution may be represented as

$$\begin{aligned} w_{k+1}(x_1, x_2) &= w_k(x_1, x_2) + \theta_k(x_1, x_2) \\ \Phi_{k+1}(x_1, x_2) &= \Phi_k(x_1, x_2) + \varphi_k(x_1, x_2). \end{aligned} \tag{6.1}$$

Substituting now expressions (6.1) in equations (4.5) and retaining only linear terms in the increments $\theta_k(x_1, x_2)$ and $\varphi_k(x_1, x_2)$, we arrive at the system of equations

$$\begin{aligned} &D\nabla^2\nabla^2\theta_k - h\sigma_i^0\Lambda\theta_k - L(\theta_k, \Phi_k) - L(w_k, \varphi_k) - k_{11}\varphi_{k,22} - k_{22}\varphi_{k,11} - \Delta\mathscr{J} = 0 \\ &\nabla^2\nabla^2\varphi_k + \int_{\tau_k}^{\tau_{k+1}} [\nabla^2\nabla^2 + (m-1)\Lambda_1^2]\Phi \, d\xi = -Eh[k_{11}\theta_{k,22} + k_{22}\theta_{k,11} + h(w_k, \theta_k)] \\ &\Delta\mathscr{J} = D[e^{-\tau_k}\int_{\tau_k}^{\tau_{k+1}} e^{\xi}(\nabla^2\nabla^2 - \tfrac{3}{4}\Lambda^2)w \, d\xi + (e^{-\tau_{k+1}} - e^{-\tau_k})\int_0^{\tau_{k+1}} e^{\xi}(\nabla^2\nabla^2 - \tfrac{3}{4}\Lambda^2)w \, d\xi \\ &\quad + \frac{3mD}{4}[e^{-m\tau_k}\int_{\tau_k}^{\tau_{k+1}} e^{\xi m}\Lambda^2 w \, d\xi + (e^{-m\tau_{k+1}} - e^{-m\tau_k})\int_0^{\tau_{k+1}} e^{m\xi}\Lambda^2 w \, d\xi]. \end{aligned} \tag{6.2}$$

The functions w and Φ under the integral sign are not known in the intervals $\tau_i < \tau < \tau_{i+1}$. It is possible, however, to use the known approximate formulas of numerical integration to evaluate the integrals since the values of these functions and their first derivatives are known at $\tau = \tau_i (i = 1, \dots k)$. Hence, the process of constructing the solutions reduces to the successive integration of systems of linear differential equations of the form of (6.2) at $k = 1, 2, 3, \dots$. The non-linear equations (4.5) are solved only once for the initial instant of time $\tau_0 = 0$, which is needed to determine the initial values $w(x_1, x_2, 0)$ and $\Phi(x_1, x_2, 0)$. The solution for the kth instant of time is represented as the sum

$$\begin{aligned} w_k(x_1, x_2) &= w(x_1, x_2, 0) + \sum_{i=0}^{k-1} \theta_i(x_1, x_2) \\ \Phi_k(x_1, x_2) &= \Phi(x_1, x_2, 0) + \sum_{i=0}^{k-1} \varphi_i(x_1, x_2). \end{aligned} \tag{6.3}$$

The method of successive linearization of equations seems natural for creep problems and widens to a considerable extent the possibilities of solving non-linear problems.

7. A CIRCULAR CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

Consider the problem of stability of a circular cylindrical shell of radius R subjected to compressive forces T. The basic state of stress of such a shell is a membrane state ($\sigma_{11}^0 = -T/h; \sigma_{12}^0 = \sigma_{22}^0 = 0$) and the bending of the shell at the moment of load application and during creep occurs only because of the presence of initial imperfections in the middle surface shape which can be taken account of by assigning an initial deflection w^0. As a simple illustration of the foregoing statement of the problem and the method of solution consider the case of an axisymmetric initial deflection of the form

$$w^0 = h\zeta^0(1-\cos 2\pi x/l) \tag{7.1}$$

where x is the longitudinal co-ordinate, l is the wave length of the initial deflection, ζ^0 is the amplitude divided by the thickness.

In the case of an axisymmetric initial deflection the non-linear system of equations (4.5), which will be used here to determine stresses and strains during creep, admits of a simple axisymmetric solution

$$\begin{aligned} w(x,\tau) &= f_0(\tau) - h\zeta(\tau)\cos\frac{2\pi x}{l} \\ \Phi(x,\tau) &= -Eh^3\psi(\tau)\cos\frac{2\pi x}{l}. \end{aligned} \tag{7.2}$$

The free term $f_0(\tau)$ is determined from the single-valuedness condition for displacements in the circumferential direction, while the coefficients $\zeta(\tau)$ and $\psi(\tau)$ are determined by solving the system of equations

$$\begin{aligned} \psi(\tau)+\frac{m+3}{4}\int_0^\tau \psi(\xi)\,\mathrm{d}\xi &= \frac{1}{4\omega^2}(\dot{\zeta}-\zeta^0)\sigma_*(\zeta-\zeta^0)-\sigma\zeta = \frac{\omega^2}{g}\int_0^\tau \mathrm{e}^{\xi-\tau}(\zeta-\zeta^0)\,\mathrm{d}\xi \\ &+\frac{m\omega^2}{3}\int_0^\tau \mathrm{e}^{m(\xi-\tau)}(\zeta-\zeta^0)\,\mathrm{d}\xi+\frac{m+3}{16\omega^2}\int_0^\tau \mathrm{e}^{[(m+3)/4](\xi-\tau)}(\zeta-\zeta^0)\,\mathrm{d}\xi \end{aligned} \tag{7.3}$$

$$\sigma = \frac{TR}{Eh^2}; \qquad \omega^2 = \frac{\pi^2 Rh}{l^2}; \qquad \sigma_* = \frac{4\omega^2}{g}+\frac{1}{4\omega^2}.$$

Solution (7.2) satisfies the condition that the ends are simply supported by rings whose points may undergo radial displacements. The fact that the rings are free to expand in the radial direction is immaterial as the violation of this condition leads to an alteration in stress distribution only near the ends of the shell and the influence of these disturbances cannot obviously be compared with the influence of the periodic imperfections considered in the present case.

As can be seen, the actual state of stress of a shell having initial deflections departs from the basic one as a result of the occurrence of additional forces $T_{22}(x,\tau)(T_{22} = \Phi_{,11})$

$$T_{22} = \frac{4\pi^2 Eh^3}{l^2}\psi(\tau)\cos\frac{2\pi x}{l}. \tag{7.4}$$

The occurrence of additional forces has the effect that the original axisymmetric configuration of equilibrium becomes unstable at a certain instant of time $\tau = \tau^*$ and is transformed into an asymmetric configuration with the formation of typical buckles in the circumferential direction. Since the state of stress is not homogeneous along the length of the shell, the asymmetric buckling will be accentuated in regions of compressive circumferential forces and restrained in regions of tensile circumferential forces. One may therefore expect the existence of asymmetric configurations of equilibrium with nodal lines at points of maximum tensile forces. The axial period for such configurations of equilibrium is twice as large as the period for the axisymmetric configuration (7.2). Assuming the integration of equations (5.1) by the Bubnov method, we limit our consideration to the following expression for the deflection function w^*

$$w^*(x, y) = h\zeta^* \sin\frac{\pi x}{l} \cos\frac{n}{R}y. \tag{7.5}$$

Here n is a parameter defining the number of waves in the circumferential direction for asymmetric buckling. Determining the force function Φ^* from the second equation of system (5.1) and integrating the first equation by the Bubnov method,† we arrive at the following condition for the non-triviality of the solution of the homogeneous system of equations (5.1)

$$\sigma_{**} - \sigma - 2\eta\psi(\tau) - a\zeta(\tau) + b\zeta^2(\tau) = 0. \tag{7.6}$$

Here the following notations are introduced

$$\begin{aligned} \sigma_{**} &= \frac{\omega^2}{(\omega^2+\eta^2)^2} + \frac{(\omega^2+\eta^2)^2}{9\omega^2}; \\ a &= 4\eta^2 \frac{\omega^2}{(\omega^2+\eta^2)^2}; \qquad \eta^2 = \frac{hn^2}{R} \\ b &= 4\eta^4\omega^2\left[\frac{1}{(\omega^2+\eta^2)^2} + \frac{1}{(9\omega^2+\eta^2)^2}\right]. \end{aligned} \tag{7.7}$$

† This method was proposed by the well-known Russian shipbuilder and scientist Ivan Grigorjevich Bubnov (1872–1919) in 1913 in connection with the review of the Timoshenko's monograph, which was published by S. P. Timoshenko in 1912 in Kiev and which was awarded the D. M. Jurawskii prize (I. G. Bubnov's Review of Prof. S. P. Timoshenko's work "On the stability of elastic systems", Reports S. Petersburg Institute of the ways of communication, 1913, issue 81, pp. 33–36. Reprint: I. G. Bubnov, The selected works, Sudpromgiz, Leningrad, 1956, pp. 136–139).

In his review I. G. Bubnov indicates that it is possible, in contrast to Timoshenko's method and without resorting to the consideration of the energy of the system, to construct a simple solution by introducing a series expansion of the variable, in the total differential equation of the problem, by multiplying the resulting expression by a term of the series expansion and integrating over the whole volume of the body. At the same time he gives the solution of the problem of stability of a centrally compressed cantilever, showing that in considering this case the results, on the basis of the energy method given by S. P. Timoshenko and by this method, coincide.

One year later I. G. Bubnov solved by his own method two difficult, for his time, problems of stability: the problem of a simply supported rectangular plate in the presence of pure shear and the problem of a simply supported rectangular plate under eccentric one-sided compression (see J. G. Bubnov, Structural mechanics of the ship, S.-Peterburg, Part 2, Printing-house of marine Admiralty, 1914, §22, pp. 515–544).

Further examples of Bubnov's approximate method can be found in B. G. Galerkin's paper (B. G. Galerkin, Bars and plates. Series in certain equations of elastic equilibrium of bars and plates. Vestnik inzhenerov, 1915, 1 October, vol. 1, N 19, pp. 897–908. Reprint: B. G. Galerkin, Collected works. Edition Academy of Sciences of the USSR, Moscow, 1952, pp. 168–195). Outside of Russia Bubnov's works probably were not known. At the First International Congress for Applied Mechanics, C. B. Biezeno paid attention to the Galerkin paper (C. B. Biezeno, Graphical and numerical methods for solving stress problems. Proceedings of the First International Congress for Applied Mechanics, Delft, 1924, Technische Boekhandel en Drukkerij. J. Waltman, Jr, 1925, pp. 3–17). In connection with this Bubnov's method received abroad the name of Galerkin's method, later Bubnov–Galerkin's method.

The functions $\zeta(\tau)$ and $\psi(\tau)$ are determined by the solution of the system of equations (7.3). Equation (7.6) determines the moment of bifurcation buckling τ^*. Figures 2 and 3

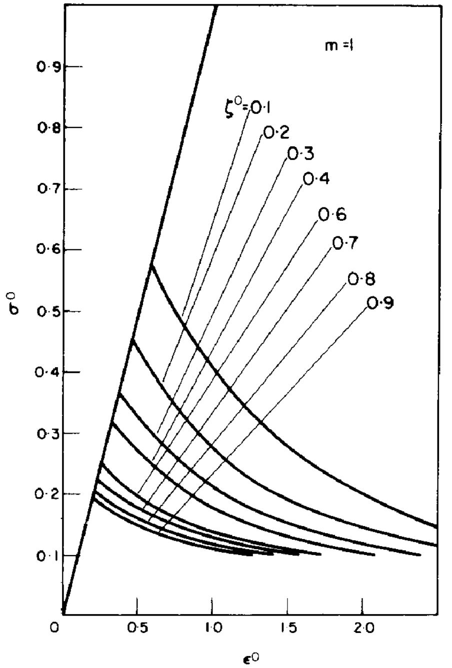

FIG. 2

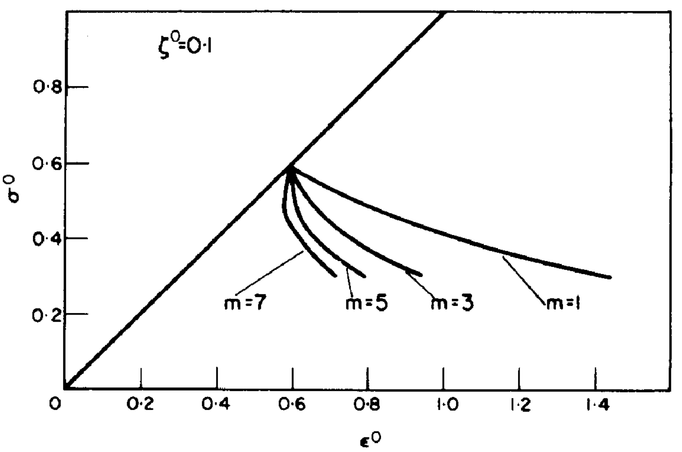

FIG. 3

present curves in which the axial critical strain ε^0 is plotted against the load parameter σ^0 for various values of the parameter ζ^0 and the creep index m. The parameter of the total critical strain ε^0 is related to the parameter τ^* by the following expression

$$\varepsilon^0 = \sigma^0(1+\tau^*). \tag{7.8}$$

Here σ^0 is the ratio of the load parameter σ to the parameter of the critical load σ_{cl} corresponding to a perfectly smooth elastic shell ($\sigma^0 = \sigma/\sigma_{cl}$). When $\nu = \frac{1}{2}$ (ν is Poisson's ratio) the parameter σ_{cl} $\{\sigma_{cl} = 1/\sqrt{[3(1-\nu^2)]}\}$ is $\frac{2}{3}$. The points of intersection of the curves and the diagonal correspond to the elastic buckling at the moment of loading ($\tau^* = 0$, $\varepsilon^0 = \sigma^0$). The relation between the critical load parameter σ^0_{cr} for an elastic shell and the parameter ζ^0 now follows as a special case. This relation is represented in Fig. 4.

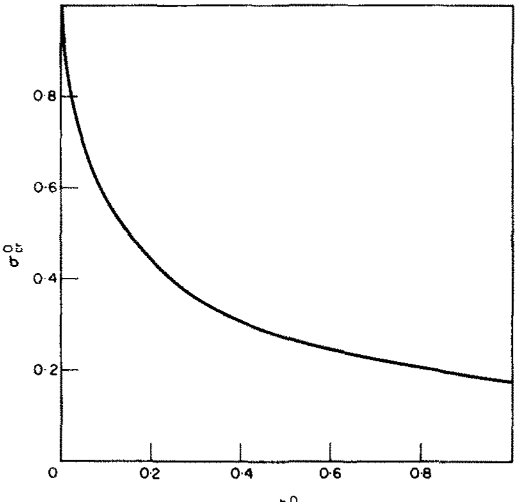

FIG. 4

Emphasizing the necessity for the investigation of bifurcation buckling of shells during creep, we shall discuss the statement of the problem developed by Hoff in [5]. According to Hoff's method the critical time is determined in relation to the initial deflection as the instant at which deflections become infinite. The important point here is the consideration of physical non-linearity. As an example we take the cases $m = 3$ and $m = 1$. To determine the nature of development of deflections in time we solve the variational equation (3.1), where the functional $\mathscr{J}$ is expressed by (3.9). The solution will be sought in the form (see also [12])

$$\begin{aligned} w(x,\tau) &= f_0(\tau) - h\zeta(\tau)\cos 2\pi x/l \\ F(x,\tau) &= -\tfrac{1}{2}Ty^2 - Eh^3\psi(\tau)\cos 2\pi x/l \\ M_{11}(x,\tau) &= m_{11}(\tau)\cos 2\pi x/l \\ M_{22}(x,\tau) &= m_{22}(\tau)\cos 2\pi x/l. \end{aligned} \tag{7.9}$$

Performing now all the integrations necessary for the determination of the functional $\mathscr{J}$ and setting the variation of the expression so obtained equal to zero, we arrive at a non-linear system of equations for the determination of time functions $\zeta(\tau)$, $\psi(\tau)$, $m_{11}(\tau)$, $m_{22}(\tau)$. Figure 5 represents the results of solution of this system of equations in the form of curves showing the amplitude of deflection $\zeta(\tau)$ in relation to time for two values of the load parameter σ^0 and the creep indices $m = 3$ and $m = 1$. As can be seen from these graphs, for $m = 3$ the amplitude of deflection increases indefinitely as the time parameter tends to a certain finite value τ^{**}, where τ^{**} depends on the amplitude ζ^0. According to Hoff, the parameter τ^{**} is considered as the critical time. For $m = 1$, however, deflections increase exponentially and the critical time does not exist in the above sense ($\tau^{**} = \infty$).

From the foregoing analysis, however, it follows that there exists a finite time $\tau = \tau^*$ when the original axisymmetric configuration of equilibrium becomes unstable and asymmetric buckling occurs. The corresponding instants of time are marked by small circles on the curves represented in Fig. 5.

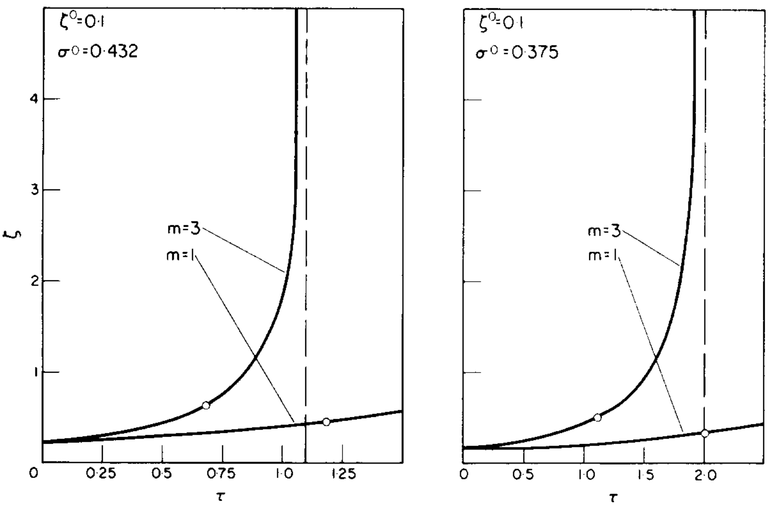

FIG. 5

8. A CIRCULAR CYLINDRICAL SHELL UNDER A LINE RADIAL LOAD AROUND A CIRCUMFERENCE

In the preceding section we have considered an example when the variation of the state of stress during creep is due to the presence of the initial deflection and the process of development of additional forces is so pronounced that at a certain instant of time $\tau = \tau^*$ the original configuration of equilibrium becomes unstable. Let us now consider an example when the variation of the state of stress and strain in time is due to the conditions of loading, the nature of deformation of a shell being such that the initial deflections can be neglected.

Let us make an analysis of stability of equilibrium in time of a long cylindrical shell of radius R loaded circumferentially with radial forces of intensity T(see Fig. 6). In this case the state of stress and strain in the shell is axisymmetric during creep up to the moment of bifurcation buckling and the system of non-linear equations (4.5) with $w^0 = 0$ admits of a simple axisymmetric solution $w = w(x, \tau)$; $\Phi = \Phi(x, \tau)$. For the creep index $m = 1$, equations (4.5) reduce to one equation

$$K\left[D\frac{\partial^4 w}{\partial x^4}+\frac{Eh}{R^2}w\right] = 0. \tag{8.1}$$

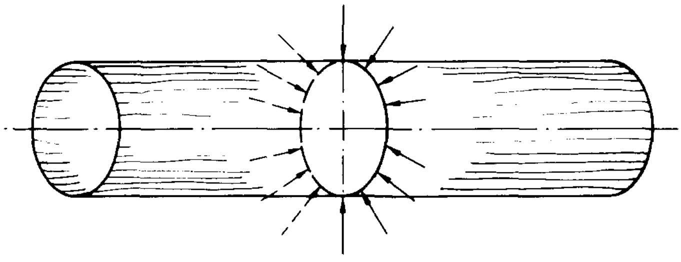

FIG. 6

At the middle section (at $x = 0$) the following conditions must be fulfilled (Q_x is the shearing force)

$$Q_x = -\frac{Eh^3}{9} K \frac{\partial^3 w}{\partial x^3} = -\frac{T}{2}; \qquad \frac{\partial w}{\partial x} = 0. \tag{8.2}$$

Here K is an integral operator such that

$$Kf(x, \tau) = f(x, \tau) - \int_0^\tau e^{\tau_0 - \tau} f(x, \tau_0)\, d\tau_0.$$

After integration with respect to time equations (8.1) and (8.2) become

$$D\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{R^2} w = 0. \tag{8.3}$$

At $x = 0$

$$\frac{\partial^3 w}{\partial x^3} = \frac{9T(1+\tau)}{2Eh^3}; \qquad \frac{\partial w}{\partial x} = 0. \tag{8.4}$$

Integrating equation (8.3) and taking into account conditions (8.4), we find

$$w(\alpha, \tau) = \frac{R\sigma(1+\tau)}{\lambda^2} a(\alpha)$$

$$w_{,\alpha\alpha} = 2R\sigma(1+\tau) b(\alpha) \tag{8.5}$$

$$T_{22} = -\frac{Eh}{R} Kw = -\frac{Eh}{\lambda^2} \sigma a(\alpha).$$

Here the following notations are introduced

$$\alpha = \frac{x}{R}; \qquad \lambda^2 = \frac{3R}{2h}; \qquad \sigma = \frac{9R^2T}{8Eh^3\lambda}$$

$$a(\alpha) = e^{-\lambda\alpha}(\sin\lambda\alpha + \cos\lambda\alpha) \tag{8.6}$$

$$b(\alpha) = e^{-\lambda\alpha}(\sin\lambda\alpha - \cos\lambda\alpha).$$

We proceed now to the investigation of the possibility of existence of asymmetric configurations of equilibrium. We represent the deflection function w^* and the force function Φ^* in the form of trigonometric series in the variable $\beta = y/R$:

$$w^*(\alpha, \beta) = \sum_{n=2}^{\infty} w_n(\alpha)\cos n\beta$$

$$\Phi^*(\alpha, \beta) = \sum_{n=2}^{\infty} \Phi_n^*(\alpha)\cos n\beta. \tag{8.7}$$

As a result the equations of neutral equilibrium are transformed to

$$\left(\frac{d^2}{d\alpha^2} - n^2\right)^2 w_n^*(\alpha) - 4n^2\lambda^2\sigma a w_n^*(\alpha) + \frac{1}{EhR}[8n^2\lambda^4\sigma(1+\tau)b\Phi_n^* - 4\lambda^4\Phi_{n,\alpha\alpha}^*] = 0$$

$$\frac{1}{EhR}\left(\frac{d^2}{d\alpha^2} - n^2\right)^2 \Phi_n^*(\alpha) + w_{n,\alpha\alpha}^* - 2n^2\sigma(1+\tau)bw_n^* = 0. \tag{8.8}$$

Since asymmetric buckling is of a local nature, it is natural to represent the solutions for $w_n^*(\alpha)$ and $\Phi_n^*(\alpha)$ in the form of Fourier integrals

$$w_n^*(\alpha) = h\left(\sqrt{\frac{2}{\pi}}\right)\int_0^\infty v(p)\cos\alpha p\,dp$$

$$\Phi_n^*(\alpha) = Eh^2R\left(\sqrt{\frac{2}{\pi}}\right)\int_0^\infty \varphi(p)\cos\alpha p\,dp. \tag{8.9}$$

By applying Fourier cosine-transformation to equations (8.8), we obtain the following equations for the transforms $v(p)$ and $\varphi(p)$

$$(p^2+n^2)^2 v(p) + 4\lambda^4 p^2\varphi(p) - \frac{8n^2\lambda\sigma}{\pi}\int_0^\infty A(s,p)v(s)\,ds - \frac{8n^2\lambda^3\sigma(1+\tau)}{\pi}\int_0^\infty B(s,p)\varphi(s)\,ds = 0$$

$$(p^2+n^2)^2\varphi(p) = p^2 v(p) - \frac{2n^2\sigma(1+\tau)}{\pi\lambda}\int_0^\infty B(s,p)v(s)\,ds \tag{8.10}$$

where

$$A(s, p) = \frac{2\lambda^4}{4\lambda^4+(s+p)^4}+\frac{2\lambda^4}{4\lambda^4+(s-p)^4};$$
$$B(s, p) = \frac{2\lambda^2(s+p)^2}{(s+p)^4+4\lambda^4}+\frac{2\lambda^2(s-p)^2}{(s-p)^4+4\lambda^4}. \tag{8.11}$$

Eliminating the function $\varphi(p)$ from the first equation of system (8.10) and omitting the term involving the factor σ, we arrive at a homogeneous Fredholm integral equation of the second kind with the symmetric kernel

$$V(p)-\sigma\int_0^\infty \Gamma(s, p)V(s)\,ds = 0 \tag{8.12}$$

where

$$V(p) = v(p)\sqrt{\left[(p^2+n^2)^2+\frac{4\lambda^4p^4}{(p^2+n^2)^2}\right]}$$
$$\Gamma(s, p) = \frac{8n^2\lambda(p^2+n^2)(s^2+n^2)\Psi(s, p)}{\pi\sqrt{\{[4\lambda^4p^4+(p^2+n^2)^4][4\lambda^4s^4+(s^2+n^2)^4]\}}} \tag{8.13}$$
$$\Psi(s, p) = A(s, p)+\lambda^2(1+\tau)B(s, p)\left[\frac{p^2}{(p^2+n^2)^2}+\frac{s^2}{(s^2+n^2)^2}\right].$$

Since the load parameter σ appears only as a factor in front of the integral in equation (8.12), it can conveniently be considered as an independent parameter, taking the time parameter τ to be given, and the critical value of the load parameter is determined for a given time interval as the smallest characteristic number of equation (8.12). Such an approach leads to the well-known formulas expressing the first characteristic number through the traces of a kernel. As a first approximation, it may be assumed that

$$\frac{1}{\sigma^2} = \int_0^\infty\int_0^\infty [\Gamma(s, p)]^2\,dp\,ds. \tag{8.14}$$

Omitting here the details concerning the evaluation of integral (8.14), we note that the basic results have been obtained by numerical integration. Figure 7 represents a graph showing the relation of the critical time τ to the load parameter σ. The point of intersection of the curve and the axis of abscissas corresponds to buckling at the moment of loading ($\tau = 0, \sigma = 0{\cdot}39$).

In conclusion we note that the basic equations in the present paper have been obtained for specific stress–strain relations. But the foregoing examples of solution of particular problems provide convincing evidence that the above statement of p. oblems of shell stability is equally acceptable for any creep law.

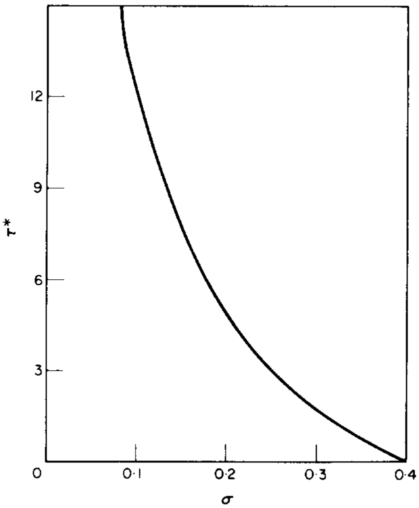

FIG. 7

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Абстракт—В данной статье рассматривается устойчивость оболочек, материал которых проявляет свойства ползучести. Считается, что после приложения внешней нагрузки оболочка не теряет устойчивости. Под действием нагрузки она изгибается. Происходит изменение во времени кривизн оболочки и усилий в ней за счет начального прогиба и ползучести материала. Этот процесс развития приводит с течением времени к неустойчивости исходной формы равновесия. Таким путем определяется критическое время. Авторы сперва находят напряженное и деформированное состояние оболочки до выпучивания, а потом исследуют возможность смены форм равновесия. Дается физически и геометрически нелинейная постановка задачи; для докритического состояния физические соотношения линеаризируются. Нелинейная система, описывающая докритическое состояние, решается шаговым методом. Система описывающая нейтральное равновесие является линейной. Общая теория проиллюстрирована для случая продольно либо радиально сжатой круговой цилиндрической оболочки. Здесь существенно использдвание метода Бубнова. Заметим, что в первой задаче критическое время является конечным и соответствует времени перехода исходной осесимметричной формы в несимметричную форму.